

**Approximate initial data for binary black holes**

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We construct approximate analytical solutions to the constraint equations of general relativity for binary black holes of arbitrary mass ratio in quasicircular orbit. We adopt the puncture method to solve the constraint equations in the transverse-traceless decomposition and consider perturbations of Schwarzschild black holes caused by boosts and the presence of a binary companion. A superposition of these two perturbations then yields approximate, but fully analytic binary black hole initial data that are accurate to first order in the inverse of the binary separation and the square of the black holes' momenta.

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**I. INTRODUCTION**

Binary black holes are among the most promising sources of gravitational radiation for the Laser Interferometer Gravitational wave Observatory (LIGO) and other gravitational wave interferometers. LIGO has recently reached its design sensitivity, making the detection of astrophysical sources of gravitational radiation a distinct possibility. To aid both in the identification of such a signal and in its interpretation, theoretical templates of potential gravitational wave forms are urgently needed.

Numerical relativity is the most promising tool for modeling the coalescence and merger of binary black holes. Typically, numerical relativity calculations compute a solution to Einstein's equations in two steps. In the first step, the constraint equations of general relativity are solved to construct initial data, describing a snapshot of the gravitational fields at one instant of time, and in the second step these initial data are evolved forward in time by solving the evolution equations.

Different approaches have been used to construct binary black hole initial data (see, e.g., the reviews [1,2] and references therein). Most earlier calculations employed the so-called transverse-traceless decomposition of the constraint equations (e.g. [3–6], see also below), while most more recent calculations solve the constraint equations in the conformal thin-sandwich formalism (e.g. [7–10]; see also [11–14] for alternative ways of constructing binary black hole initial data.) There is general consensus that the latter formalism is better suited for the construction of quasiequilibrium data (but see [15] for a very promising alternative approach), even though, at least in terms of global quantities, both formalisms lead to very similar results for configurations outside the innermost stable circular orbit (see, e.g., [9,16]).

Dynamical evolutions of binary black holes have long suffered from numerical instabilities. The past year, however, has seen dramatic progress, and at this point independent codes using very different approaches and techniques can reliably model the coalescence, merger and ring-down of binary black holes [17–21]. Some of these calculations [18,19,21] treat the black hole singularity with the so-called puncture method [22] (see also [23,24]). Given that puncture initial data for binary black holes are most easily constructed in the transverse-traceless formalism (see [4] and compare [25,26]), this development has renewed some interest in transverse-traceless initial data.

In this paper we construct black hole binary initial data for arbitrary mass ratios perturbatively. We adopt the transverse-traceless decomposition together with the puncture approach and treat both the effect of each black hole's boost as well the effect of the companion as a perturbation of a spherically symmetric Schwarzschild black hole. Both effects are axisymmetric—albeit with different axes of symmetry—allowing us to find simple analytic expressions for the gravitational fields, the location of the apparent horizon, the irreducible mass, and the total energy. To leading order we can construct binaries by simply adding the individual corrections, resulting in analytic, perturbative black hole initial data. The solution is accurate up to order  $(P/M)^2$  and  $(M/s)$ , where  $P$  is the black hole's momentum,  $M$  a measure of the black hole's mass, and  $s$  the binary separation, and becomes exact in the limit of infinite separation. We find Newtonian expressions for the energy. While this is not surprising, it is certainly reassuring. Recovering the Newtonian limit is also nontrivial, since it requires taking into account the effect of both the boost of the black holes, and the distortion due to the companion black hole.

The numerical construction of initial data for binary black holes in quasicircular orbit requires significant computational resources, since it involves stepping through

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nested iterations of elliptic solves, root-finding and sometimes even minimizations along certain sequences (see [3,4,10] and Sec. III D below). Our perturbative framework allows for a completely analytical treatment, making the construction of approximate binary black hole initial data for arbitrary mass ratios remarkably simple. Our results may therefore be of interest as initial data for dynamical simulations, especially for large binary separations where the errors are relatively small. Perhaps more importantly our results provide analytical insight into the structure of black hole binaries. Several of our intermediate results have been derived previously (e.g. [27–30]), but to the best of our knowledge they have never been combined to construct binary black hole initial data.

Our paper is organized as follows. In Sec. II we briefly review the initial value problem of general relativity and the transverse-traceless decomposition of the constraint equations, in particular. We present perturbative solutions of boosted black holes, of black holes with companions, and finally of binaries in quasicircular orbit in Sec. III. In Sec. IV we outline how the results of the previous sections can be used to construct approximate binary black hole initial data for black holes of arbitrary mass ratio in quasicircular orbit. We conclude with a discussion and summary in Sec. V. We also provide several appendices that contain all derivations and details omitted in the main body of the text. Throughout this paper we adopt geometrical units in which  $G = c = 1$ .

## II. THE INITIAL VALUE PROBLEM

Under a 3 + 1 decomposition [31,32], Einstein's equations of general relativity split into a set of constraint equations—the Hamiltonian constraint and the momentum constraint—and a set of evolution equations. Constructing a set of initial data requires specifying a spatial metric  $\gamma_{ij}$  and an extrinsic curvature  $K_{ij}$  on a spatial hypersurface  $\Sigma$  that satisfy the constraint equations. Such solutions are usually constructed with the conformal method, whose most general form is given in [33,34]. Here, we investigate a special case of the general formalism, which was known earlier, and which is amenable to analytical treatment. We introduce a conformal transformation of the spatial metric

$$\gamma_{ij} = \psi^4 \hat{\gamma}_{ij}, \quad (1)$$

where  $\psi$  is the conformal factor and  $\hat{\gamma}_{ij}$  the conformally related background metric. We also split the extrinsic curvature  $K_{ij}$  into its trace  $K$  and a conformally rescaled trace-free part  $\hat{A}_{ij}$  according to

$$K_{ij} = \psi^{-2} \hat{A}_{ij} + \frac{1}{3} \gamma_{ij} K. \quad (2)$$

In terms of these variables the Hamiltonian constraint becomes

$$8\hat{\nabla}^2 \psi - \psi \hat{R} - \frac{2}{3} \psi^5 K^2 + \psi^{-7} \hat{A}_{ij} \hat{A}^{ij} = 0, \quad (3)$$

and the momentum constraint

$$\hat{D}_j \hat{A}^{ij} - \frac{2}{3} \psi^6 \hat{\gamma}^{ij} \hat{D}_j K = 0. \quad (4)$$

Here  $\hat{\nabla}^2 = \hat{\gamma}^{ij} \hat{D}_i \hat{D}_j$  is the Laplacian,  $\hat{D}_i$  the covariant derivative, and  $\hat{R}$  the Ricci scalar associated with the background metric  $\hat{\gamma}_{ij}$ , and we have also assumed vacuum.

Both the conformal background metric  $\hat{\gamma}_{ij}$  and the trace of the extrinsic curvature  $K$  remain freely specifiable. We choose conformal flatness  $\hat{\gamma}_{ij} = f_{ij}$ , where  $f_{ij}$  is the flat metric, and maximal slicing  $K = 0$ , in which case the constraint equations reduce to

$$\hat{\nabla}^2 \psi = -\frac{1}{8} \psi^{-7} \hat{A}_{ij} \hat{A}^{ij} \quad (5)$$

and

$$\hat{D}_j \hat{A}^{ij} = 0. \quad (6)$$

Quite remarkably, the momentum constraint (6) becomes linear and decouples from the Hamiltonian constraint (5). An analytical “Bowen-York” solution to the momentum constraint, describing a black hole at coordinate location  $\mathbf{C}$  with linear momentum  $\mathbf{P}$  is given by [35–37]

$$\hat{A}_{\mathbf{CP}}^{ij} = \frac{3}{2r_C^2} [P^i n_C^j + P^j n_C^i - (f^{ij} - n_C^i n_C^j) P_k n_C^k], \quad (7)$$

where  $r_C = \|\mathbf{x}^i - \mathbf{C}^i\|$  is the coordinate distance from the center of the black hole and  $n_C^i = (\mathbf{x}^i - \mathbf{C}^i)/r_C$  is the unit vector pointing from that center to coordinate location  $\mathbf{x}$ . Given the linearity of the momentum constraint we can construct solutions for two boosted black holes by simple superposition,

$$\hat{A}^{ij} = \hat{A}_{\mathbf{C}_1 \mathbf{P}_1}^{ij} + \hat{A}_{\mathbf{C}_2 \mathbf{P}_2}^{ij}. \quad (8)$$

Equations (7) or (8) are now substituted back into the Hamiltonian constraint Eq. (5), which is then solved for the conformal factor. This step requires *boundary conditions*, which must enforce the existence of black holes in the constructed initial data. In this paper, we use the puncture method [22] (see also [23,24]) to accomplish this. In this approach, an appropriate (singular) piece is split off the conformal factor analytically, and Eq. (5) is rewritten as an elliptic equation for the remainder, which is continuous and finite throughout. Typically, this latter equation is solved numerically. We will instead construct an approximate but analytical solution perturbatively, as described in the following section.

## III. PERTURBATIVE SOLUTIONS

We construct perturbative binary black hole solutions by separately considering the effects of the boost and the companion on each black hole, and then combining the results. In Sec. III A we first consider an isolated black hole

of bare mass  $\mathcal{M}$  with boost  $\mathbf{P}$  and construct a solution that is accurate to second order in  $\epsilon_P$ , where

$$\epsilon_P \equiv \frac{P}{\mathcal{M}}. \quad (9)$$

In Sec. III B we then consider a static black hole of bare mass  $\mathcal{M}_1$  with a companion of bare mass  $\mathcal{M}_2$  at a coordinate distance  $s$  and find a solution that is accurate to first order in  $\epsilon_s$ , where

$$\epsilon_s \equiv \frac{\mathcal{M}}{s}. \quad (10)$$

In order to avoid unnecessarily complicated notation we do not distinguish between  $\mathcal{M}_1$  and  $\mathcal{M}_2$  in  $\epsilon_P$  and  $\epsilon_s$ , and instead simply point out that these expressions do depend on the mass ratio.

Finally, in Sec. III C we combine results to find perturbative binary black hole solutions. For systems in equilibrium the Virial theorem implies that we must have

$$\epsilon_s \approx \epsilon_P^2, \quad (11)$$

so that the orders of expansion for our treatment of the boost and the companion are consistent with each other.

### A. A single boosted black hole

Consider a static black hole with bare mass  $\mathcal{M}$  at coordinate location  $\mathbf{C}$ . The asymptotically flat solution to the Hamiltonian constraint (5) is then the well-known expression

$$\psi = 1 + \frac{\mathcal{M}}{2r_C}, \quad (12)$$

describing a Schwarzschild black hole in isotropic coordinates.

To generalize this solution for a boosted black hole with momentum  $\mathbf{P}$  we adopt the puncture method and split the conformal factor into two terms,

$$\psi = \frac{1}{\alpha} + u_P. \quad (13)$$

Here

$$\frac{1}{\alpha} = 1 + \frac{\mathcal{M}}{2r_C} \quad (14)$$

absorbs the singular term analytically, and  $u_P$  is a regular correction term that accounts for the effects of the boost. In terms of  $u_P$  the Hamiltonian constraint (5) becomes

$$\hat{\nabla}^2 u_P = -\beta(1 + \alpha u_P)^{-7}, \quad (15)$$

where

$$\beta = \frac{1}{8}\alpha^7 \hat{A}_{ij}^{\text{CP}} \hat{A}_{\text{CP}}^{ij}. \quad (16)$$

Since  $\hat{A}_{\text{CP}}^{ij}$  scales with  $P$ , the leading order term in (15) scales with  $\epsilon_P^2$  and all odd-order terms in  $\epsilon_P$  must vanish.

An analytical solution to order  $\epsilon_P^2$  is given by

$$u_P = \frac{\mathcal{M}\epsilon_P^2}{8(\mathcal{M} + 2r_C)^5} (u_0(r_C)P_0(\cos\theta) + u_2(r_C)P_2(\cos\theta)) + \mathcal{O}(\epsilon_P^4), \quad (17)$$

where

$$P_0(\cos\theta) = 1 \quad (18)$$

and

$$P_2(\cos\theta) = \frac{3}{2}\cos^2\theta - \frac{1}{2} \quad (19)$$

are Legendre polynomials and where the radial functions  $u_0(r_C)$  and  $u_2(r_C)$  are

$$u_0(r_C) = \mathcal{M}^4 + 10\mathcal{M}^3 r_C + 40\mathcal{M}^2 r_C^2 + 80\mathcal{M} r_C^3 + 80r_C^4, \quad (20)$$

and

$$u_2(r_C) = \frac{\mathcal{M}}{5r_C^3} \left( 42\mathcal{M}^5 r_C + 378\mathcal{M}^4 r_C^2 + 1316\mathcal{M}^3 r_C^3 + 2156\mathcal{M}^2 r_C^4 + 1536\mathcal{M} r_C^5 + 240r_C^6 + 21\mathcal{M}(\mathcal{M} + 2r_C)^5 \ln\left(\frac{\mathcal{M}}{\mathcal{M} + 2r_C}\right) \right) \quad (21)$$

(see Appendix A 1 as well as [29]). Expanding the last term of  $u_2(r_C)$  about  $r_C = 0$  shows that  $u_2(r_C) = \mathcal{O}(r_C^2)$ .

From the conformal factor (13) we can now compute several quantities of interest. A short calculation in Appendix A 4 shows that the ADM energy  $E$  of the solution is

$$E = \mathcal{M} + \frac{5}{8}\mathcal{M}\epsilon_P^2 + \mathcal{O}(\epsilon_P^4) = \mathcal{M} + \frac{5P^2}{8\mathcal{M}} + \mathcal{O}(\epsilon_P^4). \quad (22)$$

We would also like to know the irreducible mass

$$M = \sqrt{\frac{A}{16\pi}}, \quad (23)$$

where  $A$  is the proper area of the black hole's event horizon [38]. Since we cannot determine the location of the event horizon from initial data alone, we approximate this area by the area of the apparent horizon. One might expect that computing the irreducible mass to second order in  $\epsilon_P$  requires the position of the horizon to second order as well. As it turns out, however, the second order term in the horizon's position cancels out (see Appendix A 3), so that we only need the first order correction (see Appendix A 2)

$$h = \frac{\mathcal{M}}{2} - \frac{\mathcal{M}}{16} \epsilon_P \cos \theta + \mathcal{O}(\epsilon_P^2). \quad (24)$$

As we show in Appendix A 3, the irreducible mass  $M$  is then given by

$$M = \mathcal{M} + \frac{1}{8} \mathcal{M} \epsilon_P^2 + \mathcal{O}(\epsilon_P^4) = \mathcal{M} + \frac{P^2}{8\mathcal{M}} + \mathcal{O}(\epsilon_P^4). \quad (25)$$

We have verified the results for the ADM energy  $E$  (22) and irreducible mass  $M$  (25) by comparing them with numerical results obtained with a pseudospectral elliptic solver [39]. Spectral techniques are well suited for this purpose, since they provide sufficient accuracy to resolve the higher-order terms neglected in the analytic treatment. We note that the spectral elliptic solver appears to handle puncture data well, despite initial concerns about low differentiability of  $u$  at the punctures. This observation would benefit from further investigation.

In Fig. 1 we show  $E$  and  $M$  as a function of  $P$ , including both perturbative and numerical values and their difference. Since the leading order error of the perturbative results scales with  $\epsilon_P^4$  we find remarkably good agreement even for moderately large values of the momentum  $P$ . For conformally flat Bowen-York based initial data,  $P/M \approx$

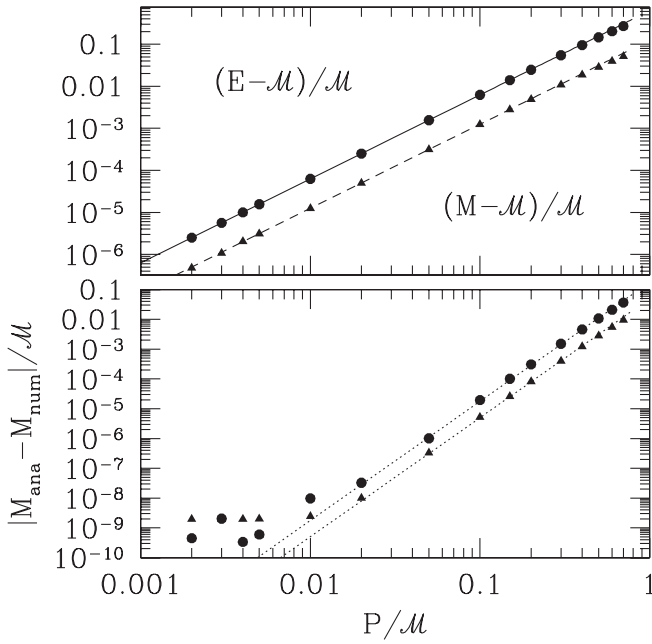


FIG. 1. The top panel shows the ADM energy (solid line) and irreducible mass (dashed) for a boosted black hole. The lines are drawn according to Eqs. (22) and (25); the points are numerical results, with error bars smaller than the points. The bottom panel shows the difference between our analytical and numerical results. The dotted lines have slopes of 4, indicating that the errors in our analytical results are of order  $\epsilon_P^4$ . The deviation from this scaling for very small values of  $P/\mathcal{M}$  is caused by the truncation error in the numerical data.

0.6 at the predicted location of the innermost stable circular orbit (ISCO, e.g. [3,4]), at which point the relative error of the perturbative values is approximately 2% for the ADM energy, and less than 1% for the irreducible mass. There is general consensus that the ISCO occurs at a somewhat larger binary separation (e.g. [10]), where both  $P$  and our errors are even smaller.

In (22) the ADM energy  $E$  is given in terms of the bare mass  $\mathcal{M}$ , which has a physical meaning only in the limit of infinite separation. It is more intuitive to express  $E$  in terms of the black hole's irreducible mass. Inverting (25) we have

$$\mathcal{M} = M - \frac{P^2}{8M} + \mathcal{O}(\epsilon_P^4), \quad (26)$$

which we can now insert into (22) to find

$$E = M + \frac{P^2}{2M} + \mathcal{O}(\epsilon_P^4). \quad (27)$$

This result is not surprising but reassuring. As expected, we can interpret the ADM energy as the sum of a “rest mass”—identified with the irreducible mass—and a Newtonian kinetic energy term. Given that we only work to order  $\epsilon_P^2$  we expect to find these Newtonian expressions only. It is worth emphasizing, however, that these

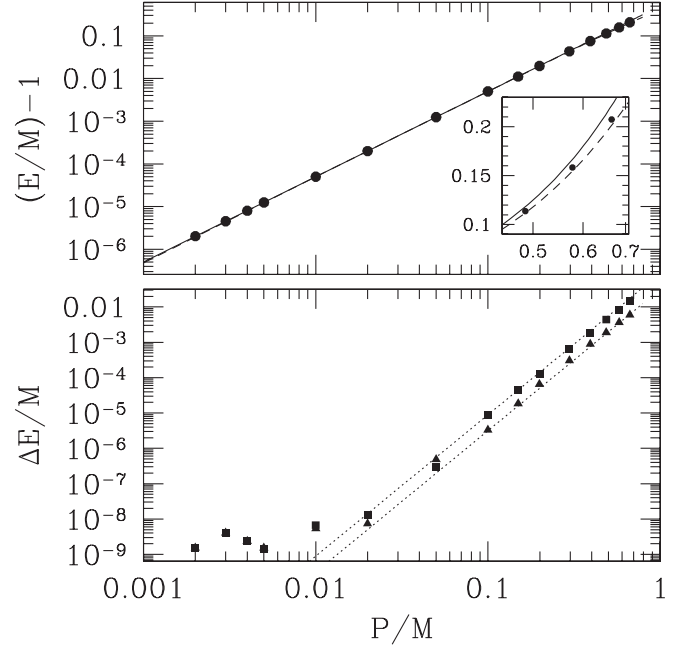


FIG. 2. The top panel shows perturbative [solid line, Eq. (27)] and numerical (dots) results for the ADM energy of a boosted black hole. The dashed line shows the ADM energy of a properly boosted Schwarzschild black hole, Eq. (28). The bottom panel shows the deviation  $(E_{\text{num}} - E_{\text{Sch}})/M$  introduced by representing the boosted black hole with conformally flat, maximally sliced puncture data (triangles) as well as the error  $|E_{\text{ana}} - E_{\text{num}}|/M$  introduced by our perturbative solution (squares). The dotted lines have slopes of 4, indicating that all errors scale as  $\epsilon_P^4$ .

Newtonian expressions emerge only after carefully taking into account the effect of the boost on the black hole's irreducible mass, which is an intrinsically relativistic object.

In Fig. 2 we show perturbative and numerical results for the ADM energy as function of  $P/M$ . We also include the special relativistic result for the ADM energy of a properly boosted Schwarzschild black hole,

$$E_{\text{Sch}} = \sqrt{M^2 + P^2} = M + \frac{P^2}{2M} - \frac{P^4}{8M^3} + \mathcal{O}(\epsilon_p^6). \quad (28)$$

Comparing the values for  $E_{\text{ana}}$ ,  $E_{\text{num}}$ , and  $E_{\text{Sch}}$  we see that the error in the ADM energy introduced by constructing transverse-traceless initial data *perturbatively* instead of exactly is only somewhat larger than the deviation of the ADM energy of conformally flat transverse-traceless puncture data from  $E_{\text{Sch}}$ . Both deviations scale as  $P^4$  and represent an excess energy over the Schwarzschild value that can be interpreted as being associated with gravitational radiation.

### B. A static black hole with a companion

We would like to derive a result as familiar as (27) for a static black hole with bare mass  $\mathcal{M}_1$  in the presence of a second static black hole with bare mass  $\mathcal{M}_2$  a coordinate distance  $s$  away. For this system, the exact solution to the Hamiltonian constraint (5) is

$$\psi = 1 + \frac{\mathcal{M}_1}{2r_{C_1}} + \frac{\mathcal{M}_2}{2r_{C_2}}. \quad (29)$$

In Eq. (29)  $r_{C_1}$  is the coordinate distance from the center of hole  $\mathcal{M}_1$ , and  $r_{C_2}$  is the coordinate distance from the center of hole  $\mathcal{M}_2$ . Evaluating the ADM energy (A33) for the conformal factor (29) yields

$$E = \mathcal{M}_1 + \mathcal{M}_2. \quad (30)$$

To calculate the irreducible masses  $M_1$  and  $M_2$  to the order  $\epsilon_s$  necessary to reproduce the expected classical result for  $E$ , we approximate  $\psi$  in a neighborhood of  $\mathcal{M}_1$  as

$$\psi = 1 + \frac{\mathcal{M}_1}{2r_{C_1}} + \frac{\mathcal{M}_2}{2s} + \mathcal{O}(\epsilon_s^2), \quad (31)$$

(see Appendix B 1). In a neighborhood of  $\mathcal{M}_2$ ,  $\psi$  is given by the same expression with the indices interchanged. We would again like to evaluate the irreducible masses  $M_1$  and  $M_2$  of the two black holes. We proceed exactly as we did for isolated boosted black holes, except that in this case we only need to expand to order  $\epsilon_s$ . Since the correction to the location of the horizon  $h$  only enters squared into the expression for the irreducible mass (see (A29) in Appendix A 3), we can completely neglect the perturbation of the horizon and may take into account only the zeroth order term

$$h = \frac{\mathcal{M}_1}{2} + \mathcal{O}(\epsilon_s). \quad (32)$$

We then find the irreducible mass

$$M_1 = \mathcal{M}_1 + \frac{\mathcal{M}_1 \mathcal{M}_2}{2s} + \mathcal{O}(\epsilon_s^2) \quad (33)$$

(see Appendix B 2). Interchanging indices yields the irreducible mass of black hole  $\mathcal{M}_2$ . In Fig. 3 we show the irreducible masses of static black holes with companions as a function of binary separation  $s$ , including both perturbative and numerical values and their difference. We again find remarkably good agreement to very small binary separations. As shown in [30], the expression (33) turns out to be accurate to at least third order in  $\epsilon_s$ , and our comparison in Fig. 3 demonstrates that the leading order error term scales with  $\epsilon_s^5$ . As computed from the transverse-traceless decomposition, for equal masses the ISCO occurs at  $s/M_{\text{tot}} \approx 2.25$  (e.g. [4]), where the relative error is approximately  $10^{-8}$ . This again sets an upper limit, since more realistically the ISCO is believed to occur at a somewhat larger binary separation.

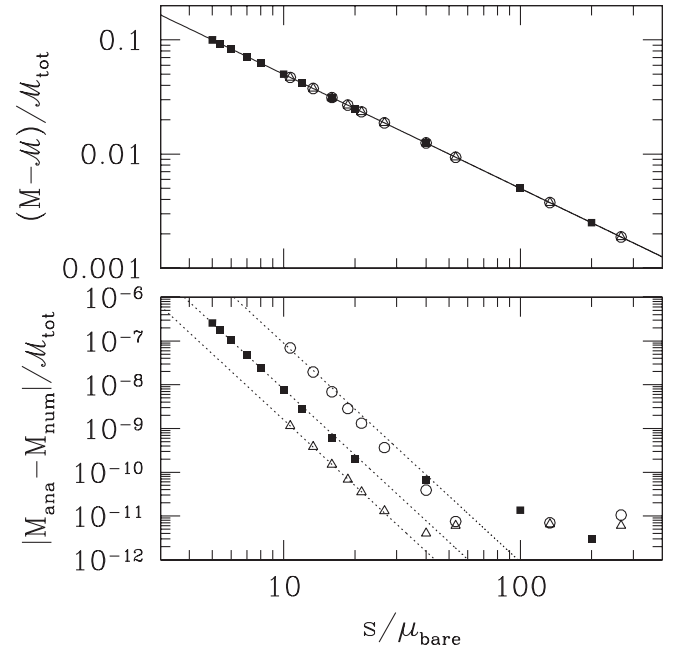


FIG. 3. The top panel shows the irreducible mass for a black hole with a companion a coordinate distance  $s$  away. Two cases with the same total bare mass  $\mathcal{M}_{\text{tot}} = \mathcal{M}_1 + \mathcal{M}_2$  but different reduced bare masses  $\mu_{\text{bare}} = \mathcal{M}_1 \mathcal{M}_2 / \mathcal{M}_{\text{tot}}$  are shown:  $\mathcal{M}_1 = \mathcal{M}_2 = 1$  and  $\mathcal{M}_1 = \frac{3}{2}$ ,  $\mathcal{M}_2 = \frac{1}{2}$ . The line drawn is the prediction of Eq. (33), and is the same for each case. The points are numerical results: squares for the larger and small masses, respectively. The bottom panel shows the difference between our analytical and numerical results. The dotted lines have slopes of  $-5$ , indicating that the errors in our analytical results are of order  $\epsilon_s^5$ .



As for the single boosted black holes we again invert (33)

$$\mathcal{M}_1 = M_1 - \frac{M_1 M_2}{2s} + \mathcal{O}(\epsilon_s^2), \quad (34)$$

and insert this expression into (30) to find the ADM energy  $E$  in terms of the irreducible mass

$$E = M_1 + M_2 - \frac{M_1 M_2}{s} + \mathcal{O}(\epsilon_s^2). \quad (35)$$

This is again the expected Newtonian result which allows us to interpret the ADM energy as the sum of the “rest mass” of the two black holes, identified with the irreducible masses, plus the potential energy.

### C. A binary black hole system

We are now in a position to construct perturbative binary black hole solutions by combining the results of Secs. III A and III B. As it turns out, we can simply add both results and obtain a solution that is accurate to order

$$\epsilon \equiv \epsilon_s \approx \epsilon_p^2 \quad (36)$$

(where we assume that  $\epsilon_s$  and  $\epsilon_p$  are similar for both black holes). More systematically, however, we proceed as follows.

First, we would like to solve the Hamiltonian constraint (5) for two black holes: one at coordinate location  $\mathbf{C}_1$  with bare mass  $\mathcal{M}_1$  and momentum  $\mathbf{P}_1$ , and the other at coordinate location  $\mathbf{C}_2$  with bare mass  $\mathcal{M}_2$  and momentum  $\mathbf{P}_2$ . As in Secs. III A and III B we start with the corresponding static solution

$$\psi = 1 + \frac{\mathcal{M}_1}{2r_{\mathbf{C}_1}} + \frac{\mathcal{M}_2}{2r_{\mathbf{C}_2}} \quad (37)$$

and find a correction  $u$  using the puncture method of [22] for nonvanishing  $\hat{A}^{ij}$ . Defining

$$\psi = u + \frac{1}{\alpha}, \quad (38)$$

with

$$\frac{1}{\alpha} = 1 + \frac{\mathcal{M}_1}{2r_{\mathbf{C}_1}} + \frac{\mathcal{M}_2}{2r_{\mathbf{C}_2}} \quad (39)$$

the Hamiltonian constraint (5) becomes a regular equation for the correction  $u$

$$\hat{\nabla}^2 u = -\beta(1 + \alpha u)^{-7}. \quad (40)$$

Here we have again used

$$\beta = \frac{1}{8}\alpha^7 \hat{A}_{ij} \hat{A}^{ij}, \quad (41)$$

and  $\hat{A}_{ij}$  is now given by Eq. (8). Equation (40) is nonlinear, and is usually solved numerically. We will solve it perturbatively, using our solutions from sections III A and III B. Intuition suggests, and Appendix C 1 proves that to fourth

order in the momenta,  $u$  is simply the sum of the boost corrections for each hole taken separately,

$$u = u_{P_1} + u_{P_2} + \mathcal{O}(\epsilon^2), \quad (42)$$

where  $u_{P_1}$  and  $u_{P_2}$  are the isolated black hole perturbations given by Eq. (17). The first neglected terms are of order  $\epsilon^2 \approx \epsilon_p^4$  because, as in Sec. III A,  $\hat{A}^{ij}$  scales with  $P$ , so that the leading order term in (40) scales with  $\epsilon_p^2$  and all odd-order terms in  $\epsilon_p$  must vanish. The conformal factor is then

$$\psi = 1 + \frac{\mathcal{M}_1}{2r_{\mathbf{C}_1}} + \frac{\mathcal{M}_2}{2r_{\mathbf{C}_2}} + u_{P_1} + u_{P_2} + \mathcal{O}(\epsilon^2), \quad (43)$$

and it follows immediately that the ADM energy of the system to this order is

$$E = \mathcal{M}_1 + \frac{5P_1^2}{8\mathcal{M}_1} + \mathcal{M}_2 + \frac{5P_2^2}{8\mathcal{M}_2} + \mathcal{O}(\epsilon^2). \quad (44)$$

Finding the irreducible mass of each hole again requires finding the apparent horizons surrounding each hole. Since we only consider the leading order perturbations of a Schwarzschild black hole, we can simply add the contributions from the boost and the companion and find

$$h_1 = \frac{\mathcal{M}_1}{2} - \frac{P_1}{16\mathcal{M}_1} \cos\theta_1 + \mathcal{O}(\epsilon), \quad (45)$$

where  $\theta_1$  is measured from  $\mathbf{P}_1$ . We note, in particular, that  $\mathcal{M}_2$ 's boost affects  $\mathcal{M}_1$  only at higher order. The irreducible mass of the black hole  $\mathcal{M}_1$  is then

$$M_1 = \mathcal{M}_1 + \frac{P_1^2}{8\mathcal{M}_1} + \frac{\mathcal{M}_1 \mathcal{M}_2}{2s} + \mathcal{O}(\epsilon^2) \quad (46)$$

(and similar for  $\mathcal{M}_2$ ; see Appendix C 2.) The correction to the irreducible mass of each black hole is the sum of the separate corrections for its boost and the presence (but not the momentum) of its companion. As before we can solve for the bare masses,

$$\mathcal{M}_1 = M_1 - \frac{P_1^2}{8M_1} - \frac{M_1 M_2}{2s} + \mathcal{O}(\epsilon^2) \quad (47)$$

and similar for  $\mathcal{M}_2$ , and express the ADM energy  $E$  in terms of the irreducible masses

$$E = M_1 + M_2 + \frac{P_1^2}{2M_1} + \frac{P_2^2}{2M_2} - \frac{M_1 M_2}{s} + \mathcal{O}(\epsilon^2). \quad (48)$$

Again, this result is not surprising but reassuring.

### D. Quasicircular orbits

Because of the circularizing effects of gravitational radiation, binary black holes at reasonably small binary separation are expected to be in approximately circular orbit. We can construct such systems by minimizing the binding energy while keeping the orbital angular momentum and irreducible masses fixed (see, e.g., [40] for an

illustration.) Since Eq. (48) represents the Newtonian energy, we will recover expressions of Newtonian orbits in what follows, but we also keep track of the error term. We define the binding energy as the difference between the ADM energy  $E$  and the black holes' irreducible masses,

$$E_b = E - M_1 - M_2. \quad (49)$$

We also define the total mass

$$M_{\text{tot}} = M_1 + M_2 \quad (50)$$

and the reduced mass

$$\mu = \frac{M_1 M_2}{M_1 + M_2}. \quad (51)$$

In a reference frame where the total momentum is zero both individual momenta  $\mathbf{P}_1$  and  $\mathbf{P}_2$  must have the same magnitude

$$P_1 = P_2 \equiv P. \quad (52)$$

In such a frame we can then write the binding energy (49) as

$$E_b = \frac{P^2}{2\mu} - \frac{M_{\text{tot}}\mu}{s} + \mathcal{O}(\epsilon^2). \quad (53)$$

Not surprisingly, this is again the Newtonian expression for the binding energy, and as a consequence the entire following discussion is essentially Newtonian.

Quasicircular orbits are those satisfying

$$\left. \frac{\partial E_b}{\partial s} \right|_{M_1, M_2, J} = 0, \quad (54)$$

where  $J = Ps$  is the orbital angular momentum (note that  $J = \mathcal{O}(\epsilon^{-1/2})$ ). In terms of  $J$ ,

$$E_b = \frac{J^2}{2s^2\mu} - \frac{M_{\text{tot}}\mu}{s} + \mathcal{O}(\epsilon^2). \quad (55)$$

Taking the derivative, we find that quasicircular orbits are those for which

$$\frac{J}{\mu M_{\text{tot}}} = \left( \frac{M_{\text{tot}}}{s} \right)^{-1/2} + \mathcal{O}(\epsilon^{1/2}), \quad (56)$$

or

$$\frac{P}{\mu} = \left( \frac{M_{\text{tot}}}{s} \right)^{1/2} + \mathcal{O}(\epsilon^{3/2}). \quad (57)$$

This condition is equivalent to a Virial relation and justifies our relation (36). Consequently, quasicircular orbits have a binding energy

$$\frac{E_{b\text{QC}}}{\mu} = -\frac{1}{2} \frac{M_{\text{tot}}}{s} + \mathcal{O}(\epsilon^2), \quad (58)$$

as one might have expected. The orbital angular frequency,  $\Omega$ , measured at infinity, is

$$\Omega = \left. \frac{\partial E_b}{\partial J} \right|_{M_1, M_2, s}. \quad (59)$$

Inserting the binding energy (55) we find

$$M_{\text{tot}}\Omega = \left( \frac{M_{\text{tot}}}{s} \right)^{3/2} + \mathcal{O}(\epsilon^{5/2}), \quad (60)$$

which we recognize as Kepler's third law. Substituting into Eq. (58), we finally find

$$\frac{E_{b\text{QC}}}{\mu} = -\frac{1}{2} (M_{\text{tot}}\Omega)^{2/3} + \mathcal{O}((M_{\text{tot}}\Omega)^{4/3}). \quad (61)$$

Evidently, our expressions for the angular momentum, the binding energy and the orbital angular frequency are, as expected, simply the Newtonian point-mass expressions. Consequently, our perturbative approach gives asymptotically correct results at sufficiently large separations. The errors in our approach are dominated by the first post-Newtonian contributions. As an illustration, Fig. 4 shows the difference between our result for the binding energy of the binary, Eq. (61), and the results of more elaborate schemes, namely, post-Newtonian expansions [41] and numerically constructed BBH sequences [6,10]. The errors in our expressions scale precisely as expected as  $(M_{\text{tot}}\Omega)^{4/3} \sim (M_{\text{tot}}/s)^2$ .

The graph demonstrates that for a given binary separation the deviations in  $E_b/\mu$  are larger than the deviations

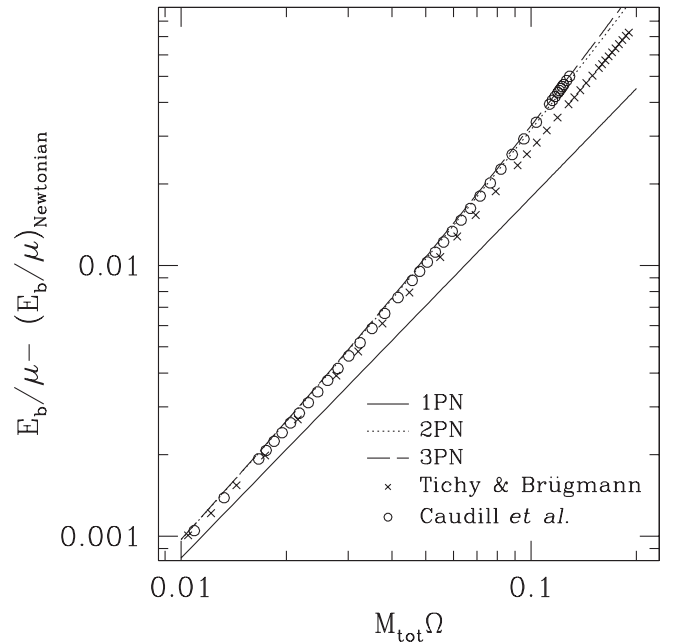


FIG. 4. The deviation in binding energy  $E_b/\mu$  of an equal mass binary between our calculation (which yields the Newtonian result) and more sophisticated approaches as a function of orbital frequency. We compare with the post-Newtonian expansion of [41] as well as numerical relativity results [6,10]. The line for “1PN” has precisely the slope of our error term,  $(M_{\text{tot}}\Omega)^{4/3}$ , confirming that our analysis finds the correct scaling.

for the individual contributions of the boost and the companion, cf. Figs. 2 and 3. This indicates that those  $\mathcal{O}(\epsilon^2)$  terms which we neglected when constructing the binary (cf. Sec. III C) have larger coefficients than those for single boosted or tidally deformed black holes.

#### IV. CONSTRUCTING INITIAL DATA

In this section we briefly describe how, for arbitrary irreducible masses  $M_1$  and  $M_2$  and binary separation  $s$ , we can use the results of the previous sections to construct approximate binary black hole initial data that are accurate to order  $\epsilon$ .

Without loss of generality we assume the orbital plane to be the  $z = 0$  plane of a Cartesian coordinate system, and assume the  $x$  axis to connect the two holes. For circular orbits the momenta  $\mathbf{P}_{1,2}$  must then be aligned with the  $y$  axis. In a reference frame where the total momentum is zero we must also have

$$\mathbf{P}_1 = -\mathbf{P}_2. \quad (62)$$

From (57) the magnitude of the momenta is given by

$$P_1 = P_2 = \frac{\mu}{\sqrt{s/M_{\text{tot}}}} + \mathcal{O}(\epsilon^2). \quad (63)$$

Using (47) we find the bare mass

$$\mathcal{M}_1 = M_1 \left( 1 - \frac{M_2^2}{8s(M_1 + M_2)} - \frac{M_2}{2s} \right) + \mathcal{O}(\epsilon^2) \quad (64)$$

and  $\mathcal{M}_2$  is given by the same expression with the indices interchanged. To place the center of (irreducible) mass at the origin of a Cartesian coordinate system, we choose

$$\mathbf{C}_1 = \left( \frac{M_2 s}{M_1 + M_2}, 0, 0 \right) \quad (65)$$

for  $\mathcal{M}_1$  and

$$\mathbf{C}_2 = \left( -\frac{M_1 s}{M_1 + M_2}, 0, 0 \right) \quad (66)$$

for  $\mathcal{M}_2$ . With the momenta and positions of the two black holes given, the extrinsic curvature  $\hat{A}_{ij}$  can now be computed from (8) and (7). Finally, the conformal factor  $\psi$  is given by Eqs. (43) and (17). This completes the construction of our approximate binary black hole initial data.

#### V. SUMMARY

We construct approximate but analytical binary black hole initial data. Adopting the puncture method (see [22–24]) we solve the constraint equations of general relativity in the transverse-traceless decomposition for perturbations of isolated, static Schwarzschild black holes, caused by a boost and the presence of a binary companion. We then use a superposition of the individual perturbations to construct analytical black hole binary solutions that are accurate to first order in the square of the momenta and the inverse of

the binary separation. In particular, our initial data become exact in the limit of infinite separation.

For binary black hole solutions constructed from the transverse-traceless decomposition of the constraint equations, quasicircular orbits are usually identified by locating a minimum of the binding energy along sequences of constant mass and constant angular momentum (see Sec. III D). For numerical solutions this requires a large number of iterations, making the problem quite involved. In contrast, for our analytical solutions this becomes very simple. In Sec. IV we describe how approximate binary black hole solutions in quasicircular orbit, for arbitrary mass ratio, can be set up analytically. Given the amount of work required to construct numerical solutions, our approximate solutions might be an attractive alternative for some applications, especially for large binary separations where our approximation becomes increasingly accurate.

Given that we expand only to first order in the square of the momentum and the inverse of the binary separation, all expressions for global quantities like the energy, angular momentum and orbital frequency take their Newtonian values. This does not mean, however, that our calculation is Newtonian. In fact, the above quantities only take the expected Newtonian values after carefully taking into account corrections to the black holes' irreducible mass, which is an intrinsically relativistic quantity. This study was partly motivated by our curiosity in the behavior of the bare mass along sequences of constant irreducible mass (compare the discussion in Ref. [42], in particular endnote [35] of that article.) Moreover, our solutions provide approximate values for the conformal factor and the extrinsic curvature together with the above global quantities, thereby specifying complete sets of initial data describing binary black holes in approximate quasicircular orbit.

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#### APPENDIX A: A BOOSTED BLACK HOLE

##### 1. The conformal factor

We would like to solve the Hamiltonian constraint (5) for a boosted black hole with bare mass  $\mathcal{M}$  and momentum  $\mathbf{P}$  located at  $\mathbf{C}$ . We follow the puncture approach of [22] and write the conformal factor  $\psi$  as

$$\psi = \frac{1}{\alpha} + u_p \quad (A1)$$



(see Sec. III A), in which case the problem reduces to solving

$$\hat{\nabla}^2 u_P = -\beta(1 + \alpha u_P)^{-7}, \quad (\text{A2})$$

for the correction term  $u_P$ . Here the coefficients  $\alpha$  and  $\beta$  are

$$\alpha = \left(1 + \frac{\mathcal{M}}{2r_C}\right)^{-1}, \quad (\text{A3})$$

and

$$\beta = \frac{1}{8} \alpha^7 \hat{A}_{ij}^{\text{CP}} \hat{A}_{\text{CP}}^{ij}. \quad (\text{A4})$$

The extrinsic curvature  $\hat{A}_{ij}^{\text{CP}}$  is given by (7) and yields

$$\hat{A}_{ij}^{\text{CP}} \hat{A}_{\text{CP}}^{ij} = \frac{9P^2}{r_C^4} \left(\frac{1}{2} + \cos^2 \theta\right), \quad (\text{A5})$$

where  $\theta$  is measured from  $\mathbf{P}$ . Since  $P$  enters into the equations only squared, all odd-order terms in  $P$  must vanish. We construct a perturbative solution that is accurate to order  $P^2$ , so that the leading order error term scales with  $P^4$ . Up to this order, (A2) reduces to

$$\hat{\nabla}^2 u_P + \beta + \mathcal{O}(\epsilon_P^4) = 0, \quad (\text{A6})$$

where  $\epsilon_P = P/\mathcal{M}$ . Because of the axisymmetry of this problem, it is reasonable to guess a solution of the form

$$u_P = \epsilon_P^2 (\tilde{u}_0(r_C) P_0(\cos \theta) + \tilde{u}_2(r_C) P_2(\cos \theta)) + \mathcal{O}(\epsilon_P^4), \quad (\text{A7})$$

where  $P_0(\cos \theta) = 1$  and  $P_2(\cos \theta) = (3\cos^2 \theta - 1)/2$  are Legendre polynomials. Solutions satisfying the boundary conditions  $u_P \rightarrow 0$  as  $r_C \rightarrow \infty$  and  $(\partial u_P / \partial r_C)|_{r_C=0} = 0$  are

$$\tilde{u}_0(r_C) = \frac{\mathcal{M}}{8(\mathcal{M} + 2r_C)^5} (\mathcal{M}^4 + 10\mathcal{M}^3 r_C + 40\mathcal{M}^2 r_C^2 + 80\mathcal{M} r_C^3 + 80r_C^4), \quad (\text{A8})$$

and

$$\begin{aligned} \tilde{u}_2(r_C) = & \frac{\mathcal{M}^2}{40r_C^3(\mathcal{M} + 2r_C)^5} \left( 42\mathcal{M}^5 r_C + 378\mathcal{M}^4 r_C^2 \right. \\ & + 1316\mathcal{M}^3 r_C^3 + 2156\mathcal{M}^2 r_C^4 + 1536\mathcal{M} r_C^5 \\ & \left. + 240r_C^6 + 21\mathcal{M}(\mathcal{M} + 2r_C)^5 \ln\left(\frac{\mathcal{M}}{\mathcal{M} + 2r_C}\right) \right) \end{aligned} \quad (\text{A9})$$

(see also [29]). In Sec. III A we have factored out terms that are common to both  $\tilde{u}_0(r_C)$  and  $\tilde{u}_2(r_C)$  [see (17)].

## 2. The apparent horizon

In axisymmetry, the apparent horizon is located at a distance  $r_C(\theta, \phi) = h(\theta)$  from a point  $\mathbf{C}$  located on the

axis of symmetry, where  $h(\theta)$  satisfies the following ordinary differential equation [43]

$$\begin{aligned} \partial_\theta \partial_\theta h = & -\Gamma_{BC}^A m_A u^B u^C - \left(\frac{ds}{d\theta}\right)^2 \gamma^{\phi\phi} \Gamma_{\phi\phi}^A m_A - (\gamma^{(2)})^{-1/2} \\ & \times \left(\frac{ds}{d\theta}\right) u^A u^B K_{AB} - (\gamma^{(2)})^{-1/2} \left(\frac{ds}{d\theta}\right)^3 \gamma^{\phi\phi} K_{\phi\phi}, \end{aligned} \quad (\text{A10})$$

subject to the boundary condition

$$\left. \frac{\partial h}{\partial \theta} \right|_{\theta=0, \pi} = 0. \quad (\text{A11})$$

In (A10) we use the vectors

$$m_i = (1, -\partial_\theta h, 0), \quad (\text{A12})$$

and

$$u^i = (\partial_\theta h, 1, 0). \quad (\text{A13})$$

We also abbreviate

$$\left(\frac{ds}{d\theta}\right)^2 = \gamma_{AB} u^A u^B, \quad (\text{A14})$$

where capital letters  $A, B$ , etc. run over the coordinates  $r_C$  and  $\theta$ , but not  $\phi$ . All coefficients are evaluated at the horizon location,  $r_C = h$ . Given our assumption of conformal flatness we also have

$$\gamma_{ij} = \psi^4 \begin{pmatrix} 1 & 0 & 0 \\ 0 & r_C^2 & 0 \\ 0 & 0 & r_C^2 \sin^2 \theta \end{pmatrix}. \quad (\text{A15})$$

The black hole's momentum enters (A10) through the extrinsic curvature  $K_{ij}$ .

It is easy to verify that for vanishing momentum  $P = 0$  the well-known isotropic horizon radius

$$h_0 = \frac{\mathcal{M}}{2} \quad (\text{A16})$$

satisfies (A10) and the boundary conditions. To find a perturbative solution for nonzero  $P$  we expand

$$h = h_0 + h_P + h_{P^2} + \mathcal{O}(\epsilon_P^3), \quad (\text{A17})$$

where  $h_P$  and  $h_{P^2}$  are corrections of first and second order in  $\epsilon_P$ , respectively.

To find the first order correction  $h_P$  we need to express all terms in (A10) up to order  $\epsilon_P$ , which yields

$$\partial_\theta \partial_\theta h = \partial_\theta \partial_\theta h_P + \mathcal{O}(\epsilon_P^2),$$

$$\Gamma_{BC}^A m_A u^B u^C = -\frac{h_P}{2} + \mathcal{O}(\epsilon_P^2),$$

$$\left(\frac{ds}{d\theta}\right)^2 \gamma^{\phi\phi} \Gamma_{\phi\phi}^A m_A = (\cot\theta) \partial_\theta h_P - \frac{h_P}{2} + \mathcal{O}(\epsilon_P^2),$$

$$\begin{aligned} (\gamma^{(2)})^{-1/2} \left(\frac{ds}{d\theta}\right) u^A u^B K_{AB} &= \psi^{-2} K_{\theta\theta} + \mathcal{O}(\epsilon_P^2) \\ &= -\frac{3P}{32} \cos\theta + \mathcal{O}(\epsilon_P^2), \\ (\gamma^{(2)})^{-1/2} \left(\frac{ds}{d\theta}\right)^3 \gamma^{\phi\phi} K_{\phi\phi} &= \psi^{-2} \sin^{-2}\theta K_{\phi\phi} + \mathcal{O}(\epsilon_P^2) \\ &= -\frac{3P}{32} \cos\theta + \mathcal{O}(\epsilon_P^2). \end{aligned} \quad (\text{A18})$$

The resulting equation for  $h_P$  is

$$\partial_\theta \partial_\theta h_P - h_P + (\cot\theta) \partial_\theta h_P - \frac{3P}{16} \cos\theta + \mathcal{O}(\epsilon_P^2) = 0. \quad (\text{A19})$$

The solution satisfying the boundary conditions is

$$h_P = -\frac{P}{16} \cos\theta + \mathcal{O}(\epsilon_P^2) \quad (\text{A20})$$

(compare [27]). One might expect that in order to find the irreducible mass to order  $\epsilon_P^2$  we also need the next order term  $h_{P^2}$ . However, as we will see in Sec. A 3, this second order terms cancels out, so that in fact we only need the first order correction  $h_P$ .

### 3. The irreducible mass

We would now like to find the irreducible mass of a boosted black hole, to second order in  $\epsilon_P$ . We approximate the irreducible mass as

$$M = \sqrt{\frac{A}{16\pi}}, \quad (\text{A21})$$

where  $A$  is the proper area of the apparent horizon of the black hole [38]. It can be shown (see, e.g., Appendix D in [44]) that

$$A = \int_0^{2\pi} \int_0^\pi \psi^4 r_C^2 \left(1 + \frac{1}{r_C^2} \left(\frac{\partial h}{\partial \theta}\right)^2\right)^{1/2} \sin\theta d\theta d\phi, \quad (\text{A22})$$

where  $\psi$  and  $r_C$  are evaluated at the horizon location,  $r_C = h$ . From the expansion (A17) of  $h$  we have

$$\left(\frac{\partial h}{\partial \theta}\right)^2 = \left(\frac{\partial h_P}{\partial \theta}\right)^2 + \mathcal{O}(\epsilon_P^3), \quad (\text{A23})$$

since  $h_0$  is spherically symmetric. The horizon area (A22) therefore splits into the two terms

$$\begin{aligned} A &= \int_0^{2\pi} \int_0^\pi \psi^4 r_C^2 \sin\theta d\theta d\phi + \frac{1}{2} \int_0^{2\pi} \int_0^\pi \psi^4 \left(\frac{\partial h_P}{\partial \theta}\right)^2 \\ &\quad \times \sin\theta d\theta d\phi + \mathcal{O}(\epsilon_P^3). \end{aligned} \quad (\text{A24})$$

We now insert the expansions

$$\psi = 1 + \frac{\mathcal{M}}{2r_C} + u_P + \mathcal{O}(\epsilon_P^4) \quad (\text{A25})$$

and

$$r_C = h = \frac{\mathcal{M}}{2} + h_P + h_{P^2} + \mathcal{O}(\epsilon_P^3). \quad (\text{A26})$$

Taking these terms to their respective powers we find

$$\psi^4 = 16 \left(1 - \frac{4h_P}{\mathcal{M}} - \frac{4h_{P^2}}{\mathcal{M}} + \frac{14h_P^2}{\mathcal{M}^2} + 2u_P\right) + \mathcal{O}(\epsilon_P^3), \quad (\text{A27})$$

and

$$r_C^2 = \frac{\mathcal{M}^2}{4} + \mathcal{M}h_P + \mathcal{M}h_{P^2} + h_P^2 + \mathcal{O}(\epsilon_P^3), \quad (\text{A28})$$

which we can now insert into (A24) to find

$$\begin{aligned} A &= 16\pi \mathcal{M}^2 + 16\pi \mathcal{M}^2 \int_0^\pi u_P \sin\theta d\theta \\ &\quad + 16\pi \int_0^\pi h_P^2 \sin\theta d\theta \\ &\quad + \pi \int_0^\pi \psi^4 \left(\frac{\partial h_P}{\partial \theta}\right)^2 \sin\theta d\theta + \mathcal{O}(\epsilon_P^3). \end{aligned} \quad (\text{A29})$$

Quite remarkably, the second order term  $h_{P^2}$  has canceled out of this expression.

To evaluate the horizon area we insert  $u_P$  [Eq. (A7)] from Appendix A 1 and  $h_P$  [Eq. (A20)] from Appendix A 2. Since we are only working to second order in  $\epsilon_P$  it is sufficient to evaluate  $u_P$  at the unperturbed horizon location  $h_0$ ,

$$\begin{aligned} u_P &= \frac{P^2}{2560\mathcal{M}^2} (-3422 + 672 \ln 2 \\ &\quad + (11196 - 2016 \ln 2) \cos^2\theta) + \mathcal{O}(\epsilon_P^3). \end{aligned} \quad (\text{A30})$$

In yet another remarkable happenstance the unattractive log terms disappear when the integration is carried out, and we are left with

$$A = 16\pi \mathcal{M}^2 \left(1 + \frac{P^2}{4\mathcal{M}^2}\right) + \mathcal{O}(\epsilon_P^4), \quad (\text{A31})$$

and therefore

$$M = \mathcal{M} \left(1 + \frac{P^2}{8\mathcal{M}^2}\right) + \mathcal{O}(\epsilon_P^4). \quad (\text{A32})$$

Even though we have carried out this calculation only to order  $\epsilon_P^2$ , it is clear that neither the area nor the mass can depend on the sign of  $P$ , so that all odd-order terms must disappear. We have also verified this by comparing with numerical data in Fig. 1.

#### 4. The ADM energy

The ADM energy  $E$  is defined as

$$E = -\frac{1}{2\pi} \oint_{\infty} \hat{\nabla}^i \psi d^2 S_i. \quad (\text{A33})$$

According to (13) we write the conformal factor as  $\psi = 1 + \mathcal{M}/2r_C + u_P$ . The correction term  $u_P$  is given by (A7), and evaluating its leading order term as  $r_C \rightarrow \infty$  we find

$$u_P \sim \frac{5P^2}{16\mathcal{M}r_C} + \mathcal{O}(\epsilon_P^4). \quad (\text{A34})$$

The ADM energy therefore becomes

$$E = \mathcal{M} + \frac{5P^2}{8\mathcal{M}} + \mathcal{O}(\epsilon_P^4). \quad (\text{A35})$$

### APPENDIX B: A STATIC BLACK HOLE WITH A COMPANION

#### 1. The conformal factor

The *exact* solution for the conformal factor  $\psi$  describing two static black holes is given by

$$\psi = 1 + \frac{\mathcal{M}_1}{2r_{C_1}} + \frac{\mathcal{M}_2}{2r_{C_2}}. \quad (\text{B1})$$

Focusing on  $\mathcal{M}_1$  we would like to eliminate the dependence on  $r_{C_2}$ , which we can do in terms of the expansion

$$\frac{1}{r_{C_2}} = \frac{1}{s} \sum_{n=0}^{\infty} \left( \frac{r_{C_1}}{s} \right)^n P_n(\cos\theta). \quad (\text{B2})$$

Here  $\theta$  is the angle between  $\mathbf{r}_{C_1}$  and  $\mathbf{r}_{12} = \mathbf{C}_2 - \mathbf{C}_1$ . Since we are only interested in terms up to order  $\epsilon_s = \mathcal{M}/s$  it is sufficient to keep the  $n=0$  term as long as we restrict analysis to a neighborhood of  $\mathcal{M}_1$ , where  $r_{C_1} \approx \mathcal{M}_1$ . Near  $\mathcal{M}_1$  we then have

$$\frac{1}{r_{C_2}} = \frac{1}{s} + \mathcal{O}(\epsilon_s^2), \quad (\text{B3})$$

and hence

$$\psi = 1 + \frac{\mathcal{M}_1}{2r_{C_1}} + \frac{\mathcal{M}_2}{2s} + \mathcal{O}(\epsilon_s^2), \quad (\text{B4})$$

and similar for  $\mathcal{M}_2$ . Borrowing the notation for boosted black holes we denote

$$u_{C_1} = \frac{\mathcal{M}_2}{2s} + \mathcal{O}(\epsilon_s^2), \quad (\text{B5})$$

and similar for  $\mathcal{M}_2$ .

#### 2. The irreducible mass

Evidently the effect of a companion on an otherwise spherical black hole must be axisymmetric in nature. To find the irreducible mass of the perturbed black hole  $\mathcal{M}_1$  we can therefore again evaluate the integral (A29), with  $u_P$  replaced by  $u_{C_1}$  [Eq. (B5)]. As before we can expand the horizon location as

$$h = h_0 + h_s + h_{s^2} + \mathcal{O}(\epsilon_s^3). \quad (\text{B6})$$

Unlike in the case of a boosted black hole, however, we are only interested in terms up to order  $\epsilon_s$ . Since the leading order term  $h_s$  scales with  $\epsilon_s$  and enters into the irreducible mass only squared, we may neglect its contributions. The only terms that remain in (A29) are therefore the first two. Inserting (B5) we immediately find

$$A_1 = 16\pi \mathcal{M}_1^2 \left( 1 + \frac{\mathcal{M}_2}{s} \right) + \mathcal{O}(\epsilon_s^2), \quad (\text{B7})$$

and hence

$$M_1 = \mathcal{M}_1 + \frac{\mathcal{M}_1 \mathcal{M}_2}{2s} + \mathcal{O}(\epsilon_s^2). \quad (\text{B8})$$

Interchanging indices yields  $\mathcal{M}_2$ .

### APPENDIX C: A BINARY BLACK HOLE SYSTEM

#### 1. Conformal factor

We would now like to solve the equation

$$\hat{\nabla}^2 u = -\beta + \mathcal{O}(\epsilon^2) \quad (\text{C1})$$

for two boosted black holes [here  $\epsilon$  is defined in (36)]. The source term  $\beta$  is

$$\beta = \frac{1}{8} \alpha^7 \hat{A}_{ij} \hat{A}^{ij}, \quad (\text{C2})$$

where  $\alpha$  is defined in (39) and the extrinsic curvature  $\hat{A}_{ij}$  is given by (8) as the sum of the individual terms for the two black holes. Given that this source term is nonlinear, it is not immediately evident that we can simply superpose two single boosted black hole solutions

$$u = u_{P_1} + u_{P_2} + \mathcal{O}(\epsilon^2), \quad (\text{C3})$$

each of which satisfies (C1) with their individual source term, e.g.

$$\hat{\nabla}^2 u_{P_1} = -\beta_1 + \mathcal{O}(\epsilon^2). \quad (\text{C4})$$

The superposition (C3) only satisfies (C1) to the required order if

$$\beta - \beta_1 - \beta_2 = \mathcal{O}(\epsilon^2). \quad (\text{C5})$$

This, however, is indeed the case. Any point in space is at least a coordinate distance  $s/2$  separated from one of the black holes. For concreteness, consider a point that is at least a distance  $s/2$  separated from  $\mathcal{M}_2$ . Expanding  $\beta$  at this point shows

$$\beta = \beta_1 + \mathcal{O}(\epsilon^2), \quad (\text{C6})$$

and we also have  $\beta_2 = \mathcal{O}(\epsilon^5)$ . Therefore, the superposition (C3) is a solution to (C1) to the required order.

We point out that the nonlinearity of  $\beta$  prevents us from also including spin in our analysis. Allowing for nonzero spin amounts to including additional terms in the extrinsic curvature, which then lead to cross-terms in the source term (C2). Perturbative solutions describing stationary, spinning black holes exist, but because of these cross-terms they cannot simply be added to the perturbative solutions describing nonspinning, boosted black holes to construct spinning, boosted black holes.

## 2. Irreducible mass

We finally argue why the correction to the irreducible mass of a boosted black hole with a companion is the sum of the corrections for a boost and the presence of a companion. In the neighborhood of  $\mathcal{M}_1$ , for example, the conformal factor is given by (43),

$$\psi = 1 + \frac{\mathcal{M}_1}{2r_{C_1}} + u_{C_1} + u_{P_1}, \quad (\text{C7})$$

where  $u_{C_1} = \mathcal{M}_2/(2s) + \mathcal{O}(\epsilon^2)$ , and  $\mathcal{M}_1$ 's horizon is located at

$$r_{C_1} = h_1 = \frac{\mathcal{M}_1}{2} + h_P + \mathcal{O}(\epsilon). \quad (\text{C8})$$

As before, higher-order corrections to the location of the horizon drop out. We can now proceed exactly as in Appendix A 3, using the combined correction  $v = u_{C_1} + u_{P_1}$  to the conformal factor, and find

$$A_1 = 16\pi\mathcal{M}_1^2 \left( 1 + \frac{P_1^2}{4\mathcal{M}_1^2} + \frac{\mathcal{M}_2}{s} \right) + \mathcal{O}(\epsilon^2). \quad (\text{C9})$$

The irreducible mass is therefore

$$M_1 = \mathcal{M}_1 \left( 1 + \frac{P_1^2}{8\mathcal{M}_1^2} + \frac{\mathcal{M}_2}{2s} \right) + \mathcal{O}(\epsilon^2) \quad (\text{C10})$$

as expected.

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